

Final Report

**SIMPLE ELASTICITY MODELING AND FAILURE PREDICTION FOR
COMPOSITE FLEXBEAMS**

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Abstract

A simple 2D boundary element analysis, suitable for developing cost effective models for tapered composite laminates, is presented. Constant stress and displacement elements are used. Closed-form fundamental solutions are derived. Numerical results are provided for several configurations to illustrate the accuracy of the model.

Introduction

One of the major sources of failure in composite structures is delamination initiating from stress concentration sites such as ply drop or a matrix crack. In order to design a damage tolerant structure, many parameters affecting delamination must be considered. A stacking sequence with the most favorable distribution of interlaminar stresses under specified loading, needs to be selected out of a large number of candidate configurations. A significant component to achieving affordable rotorcraft composite structures is the development of simple and accurate analytical tools that provide trend information at the preliminary design stages.

The analysis of composite structures generally requires numerical modeling. The cost associated with conventional finite element simulations of a large number of candidate configurations results in a need for developing alternative cost effective models. The available engineering beam and shell theories do not allow for a reliable prediction of all key parameters such as the peel stress due to simplifying assumptions restricting the stress or strain state. Existing approximate elasticity closed-form models¹ do not accommodate laminates with tapered geometry. One important application of such laminates is composite flexbeams in hingeless and bearingless rotors. During flight, the rotor hub arm experiences centrifugal loads as well as bending in the flapping-flexure region². In order to accommodate this bending, the stiffness of the flapping-flexure region is changed by varying the thickness of the hub arm. This thickness change is accomplished by dropping internal plies in that region. A 2D finite element analysis of flexbeam laminates² requires a mesh with several thousand elements and associated number of degrees of freedom of order 10^4 .

Cost effective models for elastic analysis of tapered composite structures can be developed by applying boundary element techniques. The boundary integral equations represent a closed-form model that does not require additional differentiation to obtain a solution. Therefore, a coarse mesh and low order elements resulting in small systems of linear algebraic equations are expected to provide accurate predictions. Boundary element modeling involves discretization of the domain boundary only. This reduces the model size compared to an interior discretization at the same level. On the other hand, singular kernel functions have to be used to solve the boundary integral equations. However, this singular formulation is consistent with singular stress fields at the crack front of dissimilar ply interface.

The first boundary element solutions for plane homogeneous orthotropic elasticity problems are provided in References 3 and 4. An analysis for laminated composites in a state of strain independent of the longitudinal direction was published in Ref. 5 for the case of axial extension and in Ref. 6 for the bending case. Only laminates with straight edges were considered. A mesh with 44 elements per ply was sufficient to reach a good agreement with classical finite difference and finite element predictions for the interlaminar stresses. However, neither of these two references provides the order of boundary elements used.

The main purpose of this work is to develop a simple methodology for applying boundary element modeling to rotorcraft composite structures. A 2D boundary integral formulation for anisotropic elasticity problems is provided. The constant stress and displacement elements, the simplest order element to program, are used. The model simplicity and accuracy are first illustrated for a cross sectional analysis of an orthotropic flat laminate subjected to axial extension. Results for a plane analysis of generic tapered configurations are discussed subsequently.

Analysis

In this section, details of a 2D boundary element elasticity modeling of composites are presented. The basic relationships are derived for a laminate exhibiting a plane deformation state. The cross sectional analysis of laminated beams undergoing a 3D strain state independent of the longitudinal direction is similar but more algebraically involved and, therefore, only briefly outlined.

Boundary Integral Equations

Consider a 2D elastic domain represented by a longitudinal section of a laminate of an arbitrary geometry, undergoing a plane deformation state. Assume that the laminate consists of sections of homogeneous material such as differently oriented plies or resin pockets. A homogeneous anisotropic material sector will be generally referred to as a sublaminar --a ply or a group of plies treated as a single unit with effective properties, in the sequel. Equivalent plane elastic properties of a sublaminar can be obtained based on the classical lamination theory⁷ and stiffness or compliance tensor transformation relations. The appropriate boundary conditions at the laminate boundary and the material interfaces are specified.

For a sublaminar, the engineering strain-displacement relationships for small displacement are

$$\epsilon_{xx} = u_{,x} \quad \epsilon_{yy} = v_{,y} \quad \gamma_{xy} = u_{,y} + v_{,x} \quad (1)$$

where x and y are Cartesian coordinates and subscript commas denote partial derivatives. The following compatibility equations can be obtained from eqns (1)

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - \gamma_{xy,xy} = 0 \quad (2)$$

The stress components satisfy the following equilibrium equations

$$\sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \sigma_{xy,x} + \sigma_{yy,y} = 0 \quad (3)$$

and the constitutive relations are

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \mathbf{S}\boldsymbol{\sigma} \quad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \quad \mathbf{C} = \mathbf{S}^{-1} \quad (4)$$

where \mathbf{S} and \mathbf{C} denote the compliance and stiffness matrices, respectively.

In order to obtain the governing integral equations, consider the following two equilibrium states: the true state

$$\boldsymbol{\sigma}_1 = \mathbf{C}\boldsymbol{\varepsilon}_1 \quad (5)$$

of a sublamine subjected to the actual boundary conditions and a known fundamental state

$$\boldsymbol{\sigma}_2 = \mathbf{C}\boldsymbol{\varepsilon}_2 \quad (6)$$

of a sublamine of the same material and geometry but subjected to a different set of boundary conditions. The only restriction is that the fundamental state has to be singular. This condition ensures a non-trivial solution. The fundamental state could be defined by “cutting” the ply from a half-plane or another simple domain subjected to a concentrated force with the point of load application being on the sublamine boundary. Two independent fundamental states associated with the force vector components could be obtained.

The following identity can be obtained from eqns (5) and (6)

$$\int_{\Omega} \boldsymbol{\sigma}_1^T \boldsymbol{\varepsilon}_2 dx dy = \int_{\Omega} \boldsymbol{\sigma}_2^T \boldsymbol{\varepsilon}_1 dx dy \quad (7)$$

where Ω is the sublamine area. Substitute eqns (1) into eqn (7), integrate by parts and use eqns (3) to obtain

$$\int_{\Gamma} \mathbf{t}_1^T \mathbf{u}_2 dl = \int_{\Gamma} \mathbf{t}_2^T \mathbf{u}_1 dl \quad (8)$$

where Γ is the sublaminate boundary and

$$\mathbf{t}_i = \begin{Bmatrix} t_{xi} \\ t_{yi} \end{Bmatrix} \quad \mathbf{u}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \quad i = 1, 2 \quad (9)$$

are the traction and displacement vectors. Equation (8) expresses the reciprocity theorem. The simplest possible boundary element model, based on this equation, is developed in the following.

Discretize the ply boundary into a number of segments or elements and assume constant \mathbf{t}_i and \mathbf{u}_i in each element. Select the middle point of an element as the node at which the concentrated load vector is applied to construct the fundamental solution. For each node i , two linear algebraic equations can be obtained from eqn (8)

$$\begin{aligned} t_{x1j} \int_{\Gamma_j} u_{2i}^1 dl + t_{y1j} \int_{\Gamma_j} v_{2i}^1 dl - u_{1j} \int_{\Gamma_j} t_{x2i}^1 dl - v_{1j} \int_{\Gamma_j} t_{y2i}^1 dl &= 0 \\ t_{x1j} \int_{\Gamma_j} u_{2i}^2 dl + t_{y1j} \int_{\Gamma_j} v_{2i}^2 dl - u_{1j} \int_{\Gamma_j} t_{x2i}^2 dl - v_{1j} \int_{\Gamma_j} t_{y2i}^2 dl &= 0 \end{aligned} \quad i, j = 1, \dots, N_k \quad (10)$$

where N_k is the number of elements, and the superscript indicates the first and second fundamental solutions. Summation over the repeated index j is assumed. The total number of equations is $2 \sum_{k=1}^M N_k$ where M is the number of sublaminate. The Cauchy principal value of the integrals is implied where necessary.

There are four parameters per element to be determined: two displacements and two traction vector components. However, two of them are specified if the element belongs to the laminate boundary and four continuity conditions have to be satisfied at the sublaminate interfaces. Thus, the boundary conditions reduce the number of unknowns to the number of equations.

It is worth noting that if the tractions and displacements are assumed to change within an element according to selected shape functions, eqns (10) have to be appropriately modified, and discontinuity points such as corners and crack fronts need to be accounted for in the analysis.

For a laminated beam subjected to a 3D state of strain independent of the longitudinal direction, the cross sectional boundary integral equations can be obtained as follows.

Let Cartesian coordinates x and y be in the cross section plane and z be the longitudinal axis. Denote the constant axial strain, bending curvatures and twist rate of the laminate by ε_0 , κ_1 , κ_2 and θ . Assume these four parameters to be zero in a fundamental state. Using the same procedure as for the plane deformation, the following reciprocity relationship similar to eqn (8) can be derived

$$\begin{aligned}
\int_{\Gamma} (\mathbf{t}_1^T \mathbf{u}_2 - \mathbf{t}_2^T \mathbf{u}_1) dl &= \varepsilon_0 \int_{\Omega} \sigma_{2zz} dx dy + \kappa_1 \int_{\Omega} x \sigma_{2zz} dx dy \\
&+ \kappa_2 \int_{\Omega} y \sigma_{2zz} dx dy + \theta \int_{\Omega} \{x \sigma_{2yz} - y \sigma_{2xz}\} dx dy
\end{aligned} \tag{11}$$

where the traction and displacement vectors now have three components. The right-hand side of eqn (11) can easily be transformed to a boundary integral. For an orthotropic laminate subjected to axial extension only, with x , y and z coordinate axes parallel to the principal material directions 1, 2 and 3, respectively, the boundary element equations similar to eqns (10) are

$$\begin{aligned}
&t_{x1j} \int_{\Gamma_j} u_{2i}^1 dl + t_{y1j} \int_{\Gamma_j} v_{2i}^1 dl - u_{1j} \int_{\Gamma_j} t_{x2i}^1 dl - v_{1j} \int_{\Gamma_j} t_{y2i}^1 dl \\
&= \varepsilon_0 \int_{\Gamma} -c_{23} v_{2i}^1 dx + c_{13} u_{2i}^1 dy \\
&t_{x1j} \int_{\Gamma_j} u_{2i}^2 dl + t_{y1j} \int_{\Gamma_j} v_{2i}^2 dl - u_{1j} \int_{\Gamma_j} t_{x2i}^2 dl - v_{1j} \int_{\Gamma_j} t_{y2i}^2 dl \\
&= \varepsilon_0 \int_{\Gamma} -c_{23} v_{2i}^2 dx + c_{13} u_{2i}^2 dy
\end{aligned} \quad i, j = 1, \dots, N_k \tag{12}$$

where c_{ij} are components of the sublamine stiffness matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & 0 & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & 0 & 0 \\ & & & 0 & c_{55} & 0 \\ & & & 0 & 0 & c_{66} \end{bmatrix} \tag{13}$$

Fundamental Solutions

Define the stress function identically satisfying the equilibrium equations (3) as

$$\sigma_{xx} = F_{,yy} \quad \sigma_{yy} = F_{,xx} \quad \sigma_{xy} = -F_{,xy} \tag{14}$$

Substitute the constitutive relations (4), expressed in terms of the stress function (14), into the compatibility equation (2) to obtain the following differential equation

$$s_{11} F_{,yyyy} - 2s_{16} F_{,xyyy} + (2s_{12} + s_{66}) F_{,xxyy} - 2s_{26} F_{,xxxy} + s_{22} F_{,xxxx} = 0 \tag{15}$$

The roots of the characteristic equation

$$s_{11}\xi^4 - 2s_{16}\xi^3 + (2s_{12} + s_{66})\xi^2 - 2s_{26}\xi + s_{22} = 0 \quad (16)$$

are complex⁸

$$\xi_1 = \lambda_1 + i\mu_1 \quad \xi_2 = \lambda_2 + i\mu_2 \quad \xi_3 = \bar{\xi}_1 \quad \xi_4 = \bar{\xi}_2 \quad (17)$$

where the roots ξ_3 and ξ_4 are complex conjugate to ξ_1 and ξ_2 . Introduce new variables

$$x_i = x + \lambda_i y \quad y_i = \mu_i y \quad i = 1, 2 \quad (18)$$

The differential equation (15) can be simplified to

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right] \left[\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right] F = 0 \quad (19)$$

If the characteristic roots ξ_1 and ξ_2 are not repeated, which is the case for anisotropic materials, F is a combination of two functions, each being a solution of Laplace's equation, that is

$$F = F_1 + F_2, \quad \frac{\partial^2 F_i}{\partial x_i^2} + \frac{\partial^2 F_i}{\partial y_i^2} = 0 \quad i = 1, 2 \quad (20)$$

The fundamental solution of Laplace's equation for a fixed point is a logarithmic function of the distance from that point.

A harmonic stress function F_k is the real part of an analytic function of the complex variable $z_k = x_k + iy_k$. The derivative of this function

$$\Phi_k = \frac{dF_k}{dz_k} \quad (21)$$

is also an analytic function of z_k . The general solution for any plane anisotropic elasticity problem can be written in the following form⁸

$$\Phi = \Phi_1(x_1, y_1) + \Phi_2(x_2, y_2) \quad (22)$$

where

$$\Phi_k(x_k, y_k) = P_k(x_k, y_k) + iQ_k(x_k, y_k) \quad k = 1, 2 \quad (23)$$

are analytic functions. Functions P_k and Q_k satisfy the Cauchy-Riemann conditions

$$\frac{\partial P_k}{\partial x_k} = \frac{\partial Q_k}{\partial y_k} \quad \frac{\partial P_k}{\partial y_k} = -\frac{\partial Q_k}{\partial x_k} \quad (24)$$

A fundamental solution for Φ_k can be written as

$$\Phi_k = C_k \ln z_k = C_k \ln \rho_k e^{i\varphi_k} = C_k \ln \rho_k + iC_k \varphi_k = P_k + iQ_k \quad k = 1, 2 \quad (25)$$

where

$$\rho_k = \sqrt{x_k^2 + y_k^2} \quad \varphi_k = \arctan \frac{y_k}{x_k} \quad (26)$$

The methodology for obtaining the constants in eqns (25) is illustrated for the case of an orthotropic material. In addition to the vanishing coefficients s_{16} and s_{26} in the compliance matrix (4), the characteristic roots (17) are purely imaginary for a practical material system. Equations (18) become

$$x_k = x \quad y_k = \mu_k y \quad (28)$$

The following expressions for the displacements and stresses can be derived from eqns (1), (4), (14), (18), (20)-(24), and (28)

$$\begin{aligned} u &= p_1 P_1 + p_2 P_2 \\ v &= -q_1 Q_1 - q_2 Q_2 \\ \sigma_{xx} &= -\mu_1^2 P_{1,x} - \mu_2^2 P_{2,x} \quad k = 1, 2 \\ \sigma_{yy} &= \frac{1}{\mu_1} Q_{1,y} + \frac{1}{\mu_2} Q_{2,y} \\ \sigma_{xy} &= -P_{1,y} - P_{2,y} \end{aligned} \quad (29)$$

where

$$p_i = -s_{11}\mu_i^2 + s_{12} \quad q_i = s_{12}\mu_i - \frac{s_{22}}{\mu_i} \quad i = 1, 2 \quad (30)$$

The first condition to obtain constants C_1 and C_2 in eqns (25) is a single-valued vertical displacement

$$v(\varphi) = v(\varphi + 2\pi) \quad (31)$$

The second condition normalizes the resultant traction force in the horizontal direction in the vicinity of the origin of the coordinate system to a half-unit. The resulting expressions are

$$C_1 = \frac{1}{2\pi} \frac{q_2}{\mu_1 q_2 - \mu_2 q_1} \quad C_2 = -\frac{1}{2\pi} \frac{q_1}{\mu_1 q_2 - \mu_2 q_1} \quad (32)$$

Another fundamental solution for Φ_k is

$$\Phi_k^* = -C_k^* i \ln z_k = C_k^* \varphi_k + i C_k^* \ln \frac{1}{\rho_k} = P_k^* + i Q_k^* \quad k = 1, 2 \quad (33)$$

Constants C_1^* and C_2^* are obtained based on the conditions of a single-valued horizontal displacement

$$u(\varphi) = u(\varphi + 2\pi) \quad (34)$$

and a half-unit resultant traction force in the vertical direction in the vicinity of the origin. The resulting expressions are

$$C_1^* = -\frac{1}{2\pi} \frac{p_2}{p_1 - p_2} \quad C_2^* = \frac{1}{2\pi} \frac{p_1}{p_1 - p_2} \quad (35)$$

The expressions for the fundamental solutions are summarized in the following

$$\begin{aligned} u_k &= p_1 P_{k1} + p_2 P_{k2} \\ v_k &= -q_1 Q_{k1} - q_2 Q_{k2} \\ \sigma_{xx} &= -\mu_1^2 P_{k1,x} - \mu_2^2 P_{k2,x} \quad k = 1, 2 \\ \sigma_{yy} &= \frac{1}{\mu_1} Q_{k1,y} + \frac{1}{\mu_2} Q_{k2,y} \\ \sigma_{xy} &= -P_{k1,y} - P_{k2,y} \end{aligned} \quad (36)$$

where

$$\begin{aligned}
P_{1k} &= C_{1k} \ln \sqrt{x_k^2 + y_k^2} & Q_{1k} &= C_{1k} \arctan \frac{y_k}{x_k} \\
P_{2k} &= C_{2k} \arctan \frac{y_k}{x_k} & Q_{2k} &= C_{2k} \ln \frac{1}{\sqrt{x_k^2 + y_k^2}} \\
C_{11} &= \frac{1}{2\pi} \frac{q_2}{\mu_1 q_2 - \mu_2 q_1} & C_{12} &= -\frac{1}{2\pi} \frac{q_1}{\mu_1 q_2 - \mu_2 q_1} \\
C_{21} &= -\frac{1}{2\pi} \frac{p_2}{p_1 - p_2} & C_{22} &= \frac{1}{2\pi} \frac{p_1}{p_1 - p_2}
\end{aligned} \tag{37}$$

The same fundamental solutions can be used for the cross sectional analysis of an orthotropic laminate subjected to uniform axial extension and bending. The compliance coefficients have to be replaced by the following parameters

$$\beta_{11} = s_{11} - \frac{s_{13}^2}{s_{33}} \quad \beta_{12} = s_{12} - \frac{s_{13}s_{23}}{s_{33}} \quad \beta_{22} = s_{22} - \frac{s_{23}^2}{s_{33}} \quad \beta_{66} = s_{66} \tag{38}$$

where s_{ij} are coefficients of the compliance matrix for an orthotropic sublaminate. The fundamental solutions for the case of general laminates subjected to a 3D strain state independent of the longitudinal direction are of a similar form as for the plane case. The boundary value problem is governed by two stress functions⁸. The differential equations are transformed into three Laplace's equations and three fundamental solutions have to be obtained.

Results

In the following, the boundary element model developed in the previous section is applied to simple laminate configurations. Numerical results for the interlaminar stresses are provided.

In order to compare the model predictions with published finite element results, consider a flat [90/0]_s graphite/epoxy laminate subjected to axial extension. The laminate cross section is shown in Fig. 1 and the material properties are provided in Table 1 (Ref. 9). A quarter of the laminate was modeled due to symmetry. The tractions are assumed zero at the free edges, the normal displacement and shear stress are set to zero at the symmetry axes, and the continuity conditions for both displacements and tractions are satisfied at the ply interface. The boundary is discretized as shown in Fig. 2. In order to implicitly account for singularities, the nodes are moved at a small distance outside the elements. The displacements and tractions are assumed constant within an element as mentioned above. As a consequence, all integrals in eqns (12) can easily be calculated analytically.

The interlaminar stress predictions are compared with results from a hybrid finite element cross sectional model⁹ in Figures 3 and 4. A boundary element mesh that has only 4 elements per ply at the ply interface, with a total of 24 elements and 48 degrees of

freedom, provides accurate predictions at the nodal points. This mesh is labeled coarse mesh in Figures 3 and 4, while the fine mesh corresponds to 448 elements with 896 degrees of freedom.

Next, a generic tapered glass/epoxy plane configuration shown in Fig. 5 is considered. The equivalent plane orthotropic material properties for the ply groups are provided in Table 2 (Ref. 2). The thick end is clamped and a uniform axial displacement is applied at a two tapered length distance from this end. One half of the laminate is modeled due to symmetry. The displacements are zero at the clamped end, the tractions are zero at the upper surface, and the vertical displacement and the shear stress are zero at the midplane.

The interlaminar traction predictions at the 0/45 interface are compared with results from ABAQUS finite element analysis in Figures 6 and 7. The ABAQUS models are generated using constant plane stress four-node reduced integration continuum elements (CPS4R). The finite element mesh sizes are 480 elements with 1098 variables that include the degrees of freedom and the Lagrange multipliers, and 7712 elements with 15972 variables. The boundary element model sizes are 72 elements with 144 degrees of freedom, and 448 elements with 896 degrees of freedom. The stress discontinuity corresponds to the intersection between the uniform and the tapered sections of the laminate. A boundary element mesh with 16 elements per ply at the ply interface, resulting in a total of 72 elements with 144 degrees of freedom, is sufficient for accurate predictions. The model size can be considerably reduced if a graded mesh is used.

Conclusion

A 2D boundary element analysis for composite laminates was presented in this work. The constant stress and displacement models, corresponding to the simplest element, with a small number of degrees of freedom resulted in accurate predictions of the interlaminar stresses for the configurations considered. The results presented in this work suggest that the boundary element method could become a basis for developing simple and accurate models for the elasticity and fracture mechanics analysis of composite structures of arbitrary geometry. The accuracy of the boundary element predictions at the contact surfaces could result in the development of local models for calculating the strain energy release rate components, which will be incorporated in the finite element modeling tools. To this end, a basis for simple and accurate plane and cross sectional boundary element modeling of composites has been established. The challenge comes with the development of closed form fundamental solutions for a 3D analysis of anisotropic structures. Generalization of the Stroh formalism could result in such solutions and is a subject of future work.

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Table 1. Properties of graphite/epoxy material system⁹

$E_{33} = 20.0 \text{ Msi (137.9 GPa)}$
$E_{11} = E_{22} = 2.1 \text{ Msi (14.5 GPa)}$
$G_{13} = G_{23} = G_{12} = 0.85 \text{ Msi (5.9 GPa)}$
$\nu_{31} = \nu_{32} = \nu_{12} = 0.21$

Table 2. Orthotropic material properties²

Material	Layup	E_{11} , GPa	E_{22} , GPa	G_{12} , GPa	ν_{12}
S2/E7T1 tape	$[0]_m$	47.6	12.6	4.81	0.28
E-glass/E7T1-2 fabric	$[\pm 45]_n$	25.3	24.1	4.56	0.153

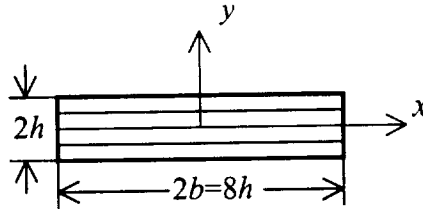


Fig. 1. Flat laminate configuration

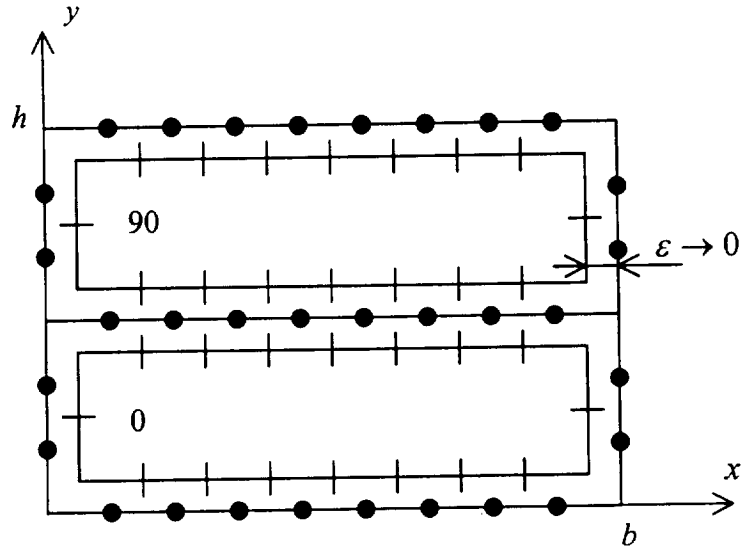


Fig. 2. Boundary discretization

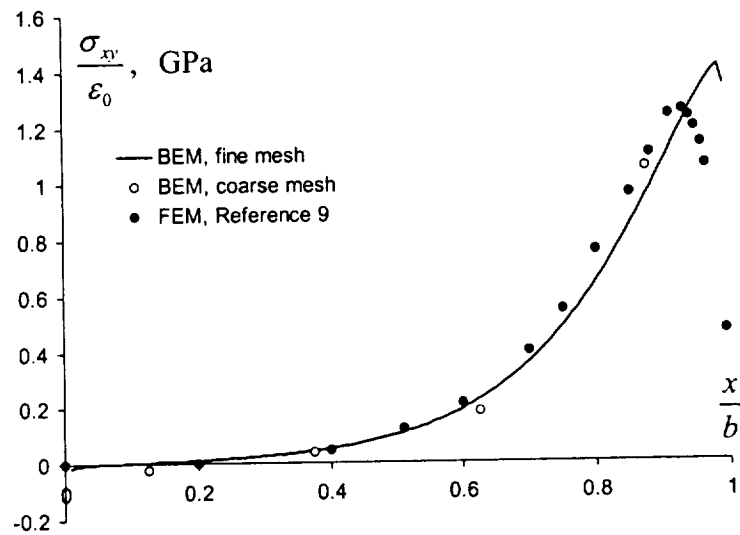


Fig. 3. Interlaminar shear stress predictions for flat laminate

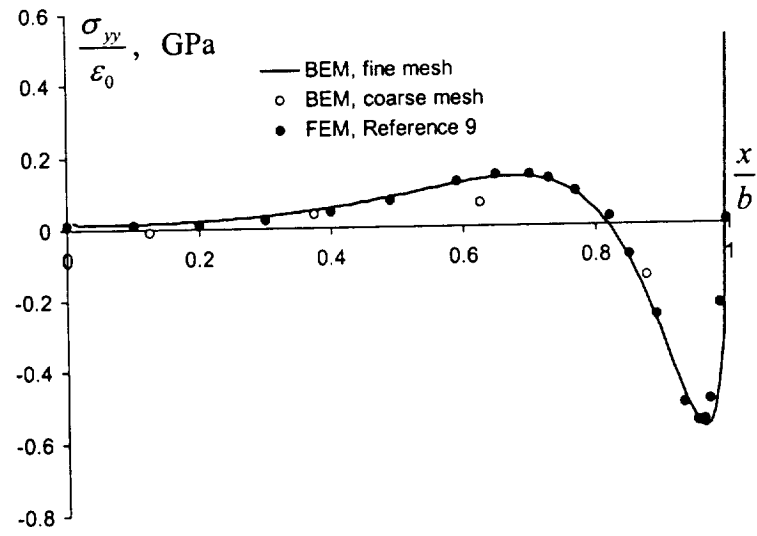


Fig. 4. Peel stress predictions for flat laminate

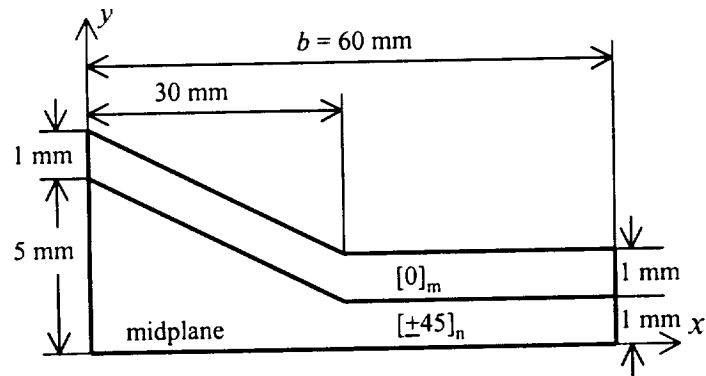


Fig. 5. Tapered laminate configuration

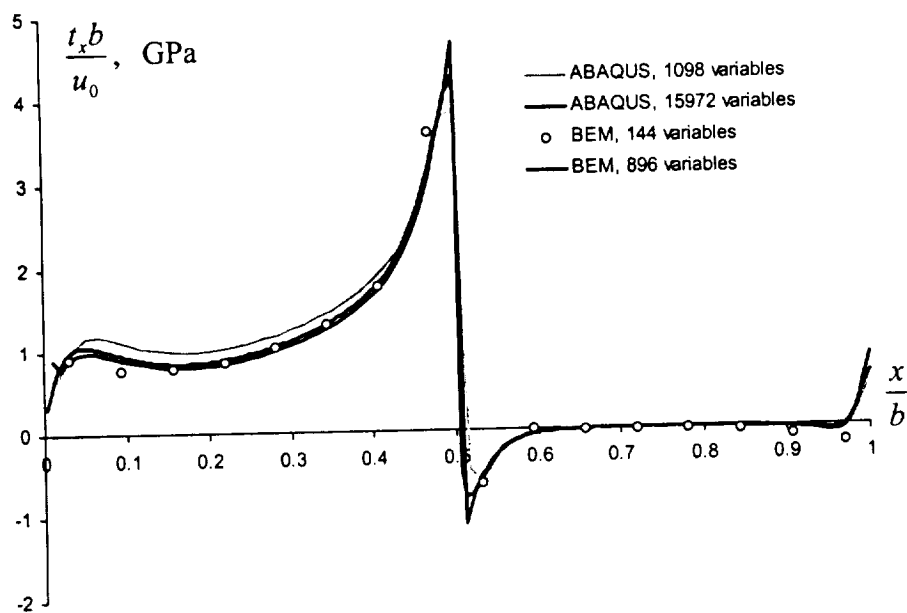


Fig. 6. Interlaminar surface traction predictions for tapered laminate

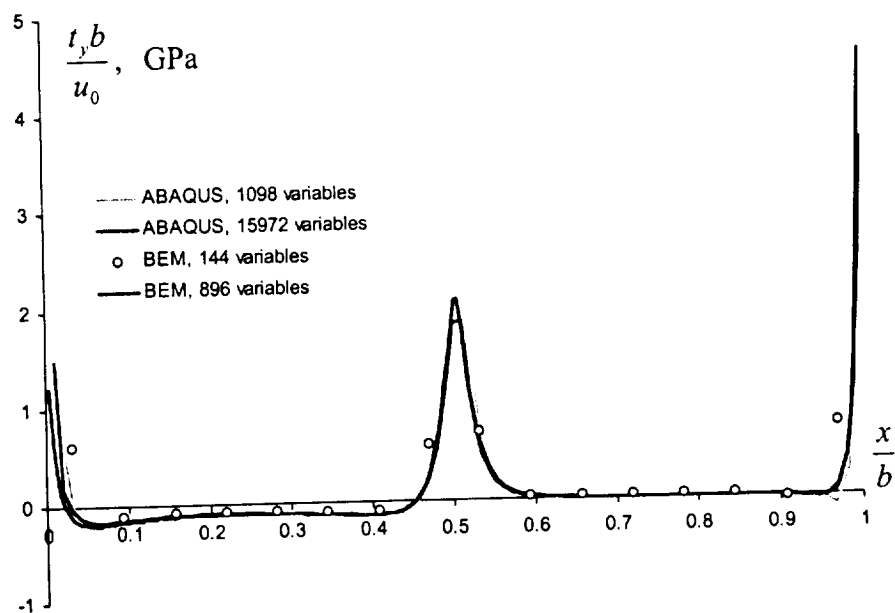


Fig. 7. Interlaminar surface traction predictions for tapered laminate